

EMBEDDINGS FOR ANISOTROPIC BESOV SPACES

F.J. PÉREZ LÁZARO

ABSTRACT. We prove embedding theorems for fully anisotropic Besov spaces. More concrete, inequalities between modulus of continuity in different metrics and of Sobolev type are obtained. Our goal is to get sharp estimates for some anisotropic cases previously unconsidered.

1. INTRODUCTION

This paper places in the theory of embeddings of spaces of differentiable functions in several variables. Our objective is to prove embeddings for anisotropic Besov spaces. The main result is a sharp embedding of different metrics (which is a generalization of the classical embedding between Nikol'skiĭ classes [9]) for Besov type spaces with all the parameters that can be different for each variable. Embeddings for anisotropic Besov spaces in Lorentz spaces are proved also. This work continues [8], where embeddings for anisotropic Sobolev spaces were found.

In order to specify better the results, let us recall an historical preview.

The study of spaces of differentiable functions in several variables with fractional index of smoothness was begun by Nikol'skiĭ, connected with problems in Approximation Theory. He obtained some embeddings for the classes $H_p^{r_1, \dots, r_n}$, characterized for Hölder conditions in L^p for the differences of the derivatives of various orders. In particular, this analogous of the theorem of Hardy-Littlewood was proved in [9]: Let $r_j > 0$ ($j = 1, \dots, n$), $r := n(\sum_{j=1}^n r_j^{-1})^{-1}$, $1 \leq p < q < \infty$. If $\varkappa := 1 - \frac{n}{r}(\frac{1}{p} - \frac{1}{q}) > 0$ and $\alpha_j := \varkappa r_j$ ($j = 1, \dots, n$), then

$$(1.1) \quad H_p^{r_1, \dots, r_n}(\mathbb{R}^n) \hookrightarrow H_q^{\alpha_1, \dots, \alpha_n}(\mathbb{R}^n).$$

Later, a theory of similar spaces was built by Besov: the scale of the so called B -spaces, introduced by him. In [2, vol.2, pg.62], using the

Thanks to V.I. Kolyada for his useful ideas and suggestions.

MSC(2000): Primary 46E35.

Key words: Besov norms, embeddings, rearrangements.

Research supported in part by grant MTM2006-13000-C03-03 of the DGI, Spain.

previous notation and $1 \leq \theta \leq \infty$, it is obtained that

$$(1.2) \quad B_{p,\theta}^{r_1,\dots,r_n}(\mathbb{R}^n) \hookrightarrow B_{q,\theta}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n).$$

Observe that in [6], Kolyada showed that (1.1) can be proved using estimates of non increasing rearrangements. Let us note that (1.1) is the particular case $\theta = \infty$ in (1.2). In fact, in [2, vol.2, pg.62], the embedding (1.2) was proved in a more general form. It is considered the case when, for each different variable x_j , is taken a different metric L^{p_j} .

It is a logical continuation to consider, for each different variable x_j , not only values p_j , but also values θ_j that can be different. However, the case when in the definition of Besov space are included different values of the parameters θ_j for each variable, had not been treated. Our objective is to find the sharp parameters for the embedding in this case. The main result in this paper is (see Theorem 3 below) an inequality that implies the embedding

$$(1.3) \quad b_{p_1,\dots,p_n;\theta_1,\dots,\theta_n}^{r_1,\dots,r_n}(\mathbb{R}^n) \hookrightarrow b_{q_1,\dots,q_n;\theta'_1,\dots,\theta'_n}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n).$$

We shall not specify here the conditions on the parameters. Let us emphasize that the new in this theorem is to obtain the optimal values θ'_j , for the case when the parameters θ_j are different.

As we shall see in Remark 1, the estimate obtained in Theorem 3 is stronger than the embedding (1.3).

In this paper we prove also Sobolev type inequalities. More specifically, we obtain embeddings of Besov spaces into Lorentz spaces. This kind of inequalities were proved and extended in [13, 11, 3, 4, 2, 6] and others. The results showed in this paper (Theorem 2 for the embedding with limit exponent and Theorem 1 for the embedding without limit exponent) extend the previous results to the case of anisotropic Besov spaces where the parameters θ'_j can be different.

Our methods are based on estimates of non increasing rearrangements. The first works using this approach on the theory of embedding of function classes are due to Ul'yanov at the end of sixties. Later, these methods were mainly developed by Kolyada (see, for instance, [5, 6, 7]). Here we use estimates of the non increasing rearrangement of a function in terms of the modulus of continuity obtained in [6]. In order to allow this approach work for anisotropic Besov spaces we need to find a kind of sharp equilibrium between the estimates. For this we apply the methods developed in [8] and used also in [12].

This paper is organized as follows. Section 2 contains the basic definitions and notations. Section 3, the lemmas used in the proofs. In

section 4, the results of this paper are presented, as well as their proofs and some remarks about them.

2. DEFINITIONS

Set $\mathbb{R}_+ \equiv (0, +\infty)$. For $1 \leq p < \infty$ we denote $\mathcal{L}^p \equiv L^p(\mathbb{R}_+, du/u)$; say also $\mathcal{L}^\infty \equiv L^\infty(\mathbb{R}_+)$ (see [4]). Note that $L^\infty(\mathbb{R}_+) = L^\infty(\mathbb{R}_+, du/u)$.

Let $S_0(\mathbb{R}^n)$ be the class of all measurable and almost everywhere finite functions f on \mathbb{R}^n such that for each $y > 0$,

$$\lambda_f(y) \equiv |\{x \in \mathbb{R}^n : |f(x)| > y\}| < \infty.$$

A non-increasing rearrangement of a function $f \in S_0(\mathbb{R}^n)$ is a non-increasing function f^* on \mathbb{R}_+ that is equimeasurable with $|f|$. The rearrangement f^* can be defined by the equality

$$f^*(t) = \sup_{|E|=t} \inf_{x \in E} |f(x)|, \quad 0 < t < \infty.$$

Assume that $0 < q, p < \infty$. A function $f \in S_0(\mathbb{R}^n)$ belongs to the Lorentz space $L^{q,p}(\mathbb{R}^n)$ if

$$\|f\|_{q,p} \equiv \left(\int_0^\infty (t^{1/q} f^*(t))^p \frac{dt}{t} \right)^{1/p} < \infty.$$

We have the inequality [1, p.217]

$$\|f\|_{q,s} \leq c \|f\|_{q,p} \quad (0 < p < s < \infty),$$

so that $L^{q,p} \subset L^{q,s}$ for $p < s$. In particular, for $0 < p \leq q$

$$L^{q,p} \subset L^{q,q} \equiv L^q.$$

The differences of degree k in the direction of the variable x_j are defined as

$$\Delta_j^k(h)f(x) \equiv \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x + ihe_j) \quad (h \in \mathbb{R}),$$

where e_k is the unit coordinate vector. The modulus of continuity in the metric $L^{q,p}$:

$$\omega_j^k(f; \delta)_{q,p} = \sup_{0 < h \leq \delta} \|\Delta_j^k(h)f\|_{q,p}.$$

As in [10, pg.152,161], we define the Besov space in the direction of the coordinate axe x_j .

Definition 1. Let $r > 0$, $1 \leq p < \infty$, $1 \leq \theta \leq \infty$ and $1 \leq j \leq n$. We define the space $B_{p,\theta;j}^r(\mathbb{R}^n)$ as the class of functions $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,\theta;j}^r} \equiv \|f\|_p + \|f\|_{b_{p,\theta;j}^r},$$

where the seminorm

$$\|f\|_{b_{p,\theta;j}^r} \equiv \left(\int_0^\infty [h^{-r} \|\Delta_j^k(h)f\|_p]^\theta \frac{dh}{h} \right)^{1/\theta} = \|h^{-r} \|\Delta_j^k(h)f\|_p\|_{\mathcal{L}^\theta}.$$

It is well known that each election of an integer $k > r$ gives equivalent seminorms. Furthermore, if we change the expression $\|\Delta_j^k(h)f\|_p$ by the modulus of continuity $\omega_j^k(f; h)_p$ we obtain equivalent seminorms also ([2, Chapter 4] and [10, Chapter 4]).

Moreover, the following inequality between seminorms holds (see, for instance [2, vol.2, pg. 64]). If $1 \leq \theta_1 < \theta_2 \leq \infty$ then $\|\cdot\|_{b_{p,\theta_2;j}^r} \leq c \|\cdot\|_{b_{p,\theta_1;j}^r}$. So, the bigger is θ , the bigger is the corresponding Besov space.

Now we are going to present anisotropic Besov spaces.

Definition 2. Let $n \in \mathbb{N}$, $r_j > 0$, $1 \leq p_j < \infty$, $1 \leq \theta_j \leq \infty$ ($j = 1, \dots, n$). We say that $f \in b_{p_1, \dots, p_n; \theta_1, \dots, \theta_n}^{r_1, \dots, r_n}(\mathbb{R}^n)$ if $f \in S_0(\mathbb{R}^n)$ and the following seminorm is finite

$$\sum_{j=1}^n \|f\|_{b_{p_j, \theta_j; j}^{r_j}}.$$

In the case $p_1 = \dots = p_n = p$ and $\theta_1 = \dots = \theta_n = \theta$ we use the notation $b_{p; \theta}^{r_1, \dots, r_n}(\mathbb{R}^n) := b_{p, \dots, p; \theta, \dots, \theta}^{r_1, \dots, r_n}(\mathbb{R}^n)$ and $B_{p; \theta}^{r_1, \dots, r_n}(\mathbb{R}^n) := L^p(\mathbb{R}^n) \cap b_{p; \theta}^{r_1, \dots, r_n}(\mathbb{R}^n)$. As usual $H_p^{r_1, \dots, r_n}(\mathbb{R}^n) := B_{p; \infty}^{r_1, \dots, r_n}(\mathbb{R}^n)$.

3. LEMMAS

The following lemma was proved in [6, pg.167]. It presents estimates of rearrangements in terms of the modulus of continuity.

Lemma 1. *Let f be a locally integrable function in $S_0(\mathbb{R}^n)$, $k_i \in \mathbb{N}$ and $p_i \in [1, \infty)$ ($i = 1, \dots, n$). Suppose that $\delta_i(t)$ ($1 \leq i \leq n$) are positive functions in \mathbb{R}_+ such that*

$$(3.1) \quad \prod_{i=1}^n \delta_i(t) = t \quad (t > 0).$$

Then, for all $0 < t < s < \infty$,

$$(3.2) \quad f^*(t) \leq (2^k - 1)f^*(s) + c \max_{i \in \{1, \dots, n\}} t^{-1/p_i} \left(\frac{s}{t} \right)^{k_i} w_i^{k_i}(f; \delta_i(t))_{p_i},$$

where $k = \max k_i$ and c is a constant which only depends on p_i and k_i .

The previous lemma is formulated in [6] for functions in $L^p(\mathbb{R}^n)$, and $p = p_1 = \dots = p_n$. If the reader checks it carefully will realize that the proof is still valid without these assumptions.

The aim of the next lemma is that given a function with some properties of monotony and integrability we can majorized it for another one with equivalent integrability properties and for which its increasing and decreasing is controlled.

Lemma 2. *Let $\alpha > 0$, $\theta \geq 1$. Let $\psi(t)$ a non negative, non decreasing function such that $t^{-\alpha}\psi(t) \in \mathcal{L}^\theta$. Then, for any $\delta > 0$ there exists a continuous and differentiable function φ on \mathbb{R}_+ such that:*

- i) $\psi(t) \leq \varphi(t)$,
- ii) $\varphi(t)t^{-\alpha-\delta}$ decreases and $\varphi(t)t^{-\alpha+\delta}$ increases,
- iii) $\|t^{-\alpha}\varphi(t)\|_{\mathcal{L}^\theta} \leq c\|t^{-\alpha}\psi(t)\|_{\mathcal{L}^\theta}$ where c is a constant that only depends on δ and α .

Proof. Follows the scheme of the one at [8, Lemma 2.1]. We include it here only for completeness.

Set

$$\varphi_1(t) = (\alpha + \delta)t^{\alpha+\delta} \int_t^\infty u^{-\alpha-\delta}\psi(u) \frac{du}{u}.$$

Then $\varphi_1(t)t^{-\alpha-\delta}$ decreases and

$$\varphi_1(t) \geq (\alpha + \delta)t^{\alpha+\delta}\psi(t) \int_t^\infty u^{-\alpha-\delta} \frac{du}{u} = \psi(t).$$

Furthermore, applying Hardy's inequality [1, pg.124], we easily get that

$$(3.3) \quad \|t^{-\alpha}\varphi_1(t)\|_{\mathcal{L}^\theta} \leq c\|t^{-\alpha}\psi(t)\|_{\mathcal{L}^\theta}.$$

Set now

$$(3.4) \quad \varphi(t) = 2\delta t^{\alpha-\delta} \int_0^t u^{-\alpha+\delta}\varphi_1(u) \frac{du}{u}$$

Then $\varphi(t)t^{-\alpha+\delta}$ increases on \mathbb{R}_+ and

$$\varphi(t) \geq \varphi_1(t) \geq \psi(t) \quad t \in \mathbb{R}_+.$$

Furthermore, the change of variable $z = u^{2\delta}$ in the right hand side of (3.4) gives that

$$\varphi(t)t^{-\alpha-\delta} = t^{-2\delta} \int_0^{t^{2\delta}} \mu(z^{1/(2\delta)}) dz,$$

where $\mu(u) = \varphi_1(u)u^{-\alpha-\delta}$ is a decreasing function on \mathbb{R}_+ . Thus, $\varphi(u)u^{-\alpha-\delta}$ decreases. Finally, using Hardy's inequality and (3.3) we get (iii). The lemma is proved. \square

Let $n \in \mathbb{N}$, $0 < r_j < \infty$, $1 \leq p_j < \infty$ and $1 \leq \theta_j \leq \infty \forall j \in \{1, \dots, n\}$. Denote

$$(3.5) \quad r = n \left(\sum_{j=1}^n \frac{1}{r_j} \right)^{-1} ; \quad p = \frac{n}{r} \left(\sum_{j=1}^n \frac{1}{p_j r_j} \right)^{-1} ; \quad \theta = \frac{n}{r} \left(\sum_{j=1}^n \frac{1}{\theta_j r_j} \right)^{-1}$$

and

$$(3.6) \quad \beta_j = \frac{1}{r_j} \left(\frac{r}{n} + \frac{1}{p_j} - \frac{1}{p} \right).$$

Then

$$(3.7) \quad \sum_{j=1}^n \beta_j = 1.$$

The equality (3.7) follows immediately from (3.5).

In the following lemma we use the notations (3.5) and (3.6).

Lemma 3. *Let $n \in \mathbb{N}$, $0 < r_j < \infty$, $1 \leq p_j < \infty$ and $1 \leq \theta_j \leq \infty$ for $j = 1, \dots, n$. Suppose that $\beta_j > 0$ for any j and let*

$$(3.8) \quad 0 < \delta \leq \frac{1}{2} \min_{1 \leq j \leq n} \{\beta_j r_j\}.$$

Let φ_j positive, strictly increasing and continuously differentiable functions on \mathbb{R}_+ , satisfying $\varphi_j(t)t^{-r_j} \in \mathcal{L}^{\theta_j}$. Besides $\varphi_j(t)t^{-r_j+\delta}$ increases and $\varphi_j(t)t^{-r_j-\delta}$ decreases. Define

$$(3.9) \quad \sigma(t) = \inf \left\{ \max_{j=1, \dots, n} \{t^{-1/p_j} \varphi_j(\delta_j)\} : \prod_{j=1}^n \delta_j = t, \delta_j > 0 \right\}.$$

Then:

i) There holds the inequality

$$(3.10) \quad \left(\int_0^\infty t^{\theta(1/p-r/n)-1} \sigma(t)^\theta dt \right)^{1/\theta} \leq c \prod_{j=1}^n [\|t^{-r_j} \varphi_j(t)\|_{\mathcal{L}^{\theta_j}}]^{\frac{r}{nr_j}}.$$

ii) There exist positive, continuously differentiable functions $\delta_j(t)$ on \mathbb{R}_+ such that

$$(3.11) \quad \prod_{j=1}^n \delta_j(t) = t$$

and

$$(3.12) \quad \sigma(t) = t^{-1/p_j} \varphi_j(\delta_j(t)) \quad (t \in \mathbb{R}_+, j = 1, \dots, n).$$

iii) for every $j = 1, \dots, n$

$$(3.13) \quad \sigma(t)t^{1/p-r/n+\delta} \uparrow \text{ and } \sigma(t)t^{1/p-r/n-\delta} \downarrow;$$

$$(3.14) \quad \delta_j(t)t^{-\beta_j/3} \uparrow \text{ and } \delta_j(t)t^{-3\beta_j} \downarrow;$$

iv) for every $j = 1, \dots, n$

$$(3.15) \quad \left(\int_0^\infty \left[\frac{\varphi_j(\delta_j(t))}{\delta_j(t)^{r_j}} \right]^{\theta_j} \frac{dt}{t} \right)^{1/\theta_j} \leq c \|t^{-r_j} \varphi_j(t)\|_{\mathcal{L}^{\theta_j}}.$$

where c depends on δ, r_j, p_j, n .

Proof. First note that

$$\lim_{\delta_j \rightarrow 0} \varphi_j(\delta_j) = 0 \quad \text{and} \quad \lim_{\delta_j \rightarrow \infty} \varphi_j(\delta_j) = \infty.$$

Now we fix $t \in \mathbb{R}_+$. It's clear that there exists a unique point $\delta_j \equiv \delta_j(t) > 0$ such that $\sigma(t) = t^{-1/p_j} \varphi_j(\delta_j(t))$. Now let $1 < \gamma < \infty$. Note that $\sigma(t) < t^{-1/p_j} \varphi_j(\delta_j(t)\gamma)$. Then

$$\sigma(t) < \min_{1 \leq j \leq n} \{t^{-1/p_j} \varphi_j(\delta_j(t)\gamma)\}.$$

By the definition (3.9) of $\sigma(t)$ we have that $\exists \delta_1^*, \dots, \delta_n^*$ so that $\prod_{j=1}^n \delta_j^* = t$ and

$$\sigma(t) \leq \max_{j=1, \dots, n} \{t^{-1/p_j} \varphi_j(\delta_j^*)\} < \min_{1 \leq j \leq n} \{t^{-1/p_j} \varphi_j(\delta_j(t)\gamma)\}.$$

Therefore

$$\delta_j^* < \delta_j(t)\gamma \quad \text{for all } j \quad \implies t = \prod_{j=1}^n \delta_j^* < \gamma^n \prod_{j=1}^n \delta_j(t).$$

Taking limits when γ tends to 1 we have $t \leq \prod_{j=1}^n \delta_j(t)$. Now if $\prod_{j=1}^n \delta_j(t) > t$ we choose $0 < \delta'_j < \delta_j(t)$ so that $\prod_{j=1}^n \delta'_j = t$. We have $\sigma(t) = t^{-1/p_j} \varphi_j(\delta_j(t)) > t^{-1/p_j} \varphi_j(\delta'_j)$. Then

$$\sigma(t) > \max_{j=1, \dots, n} \{t^{-1/p_j} \varphi_j(\delta'_j)\} \text{ and } \prod_{j=1}^n \delta'_j = t$$

which contradicts (3.9) definition of $\sigma(t)$. So functions $\delta_j(t)$ satisfy (3.11) and (3.12).

Besides, for any $j = 1, \dots, n$, by (3.12)

$$(3.16) \quad \delta_j(t) = \varphi_j^{-1}(t^{1/p_j-1/p_n} \varphi_n(\delta_n(t))).$$

Then by (3.11)

$$t = \Phi(\delta_n(t), t)$$

where

$$\Phi(s, t) = s \prod_{j=1}^{n-1} \varphi_j^{-1}(t^{1/p_j-1/p_n} \varphi_n(s))$$

which is a function of $C^1(\mathbb{R}_+^2)$ strictly increasing respect to s . In consequence, by the implicit function theorem we have that $\delta_n \in C^1(\mathbb{R}_+)$ and so, by (3.16) $\delta_j \in C^1(\mathbb{R}_+)$ for all $j = 1, \dots, n$. We have just proved (ii).

Our conditions on φ_j implies that for every $j = 1, \dots, n$

$$(3.17) \quad \frac{-r_j - \delta}{t} \leq -\frac{\varphi_j'(t)}{\varphi_j(t)} \leq \frac{-r_j + \delta}{t}.$$

Besides

$$(3.18) \quad \frac{\sigma'(t)}{\sigma(t)} = \frac{-1/p_j}{t} + \frac{\varphi_j'(\delta_j(t))}{\varphi_j(\delta_j(t))} \delta_j'(t).$$

Now we derive (3.11) and taking into account (3.7) we have that for every $t > 0 \exists m \equiv m(t)$ and $l \equiv l(t)$ such that

$$\frac{\delta_m'(t)}{\delta_m(t)} \leq \frac{\beta_m}{t} \text{ and } \frac{\delta_l'(t)}{\delta_l(t)} \geq \frac{\beta_l}{t}.$$

Then

$$\frac{-1/p_l}{t} + \frac{\varphi_l'(\delta_l(t))}{\varphi_l(\delta_l(t))} \delta_l(t) \frac{\beta_l}{t} \leq \frac{\sigma'(t)}{\sigma(t)} \leq \frac{-1/p_m}{t} + \frac{\varphi_m'(\delta_m(t))}{\varphi_m(\delta_m(t))} \delta_m(t) \frac{\beta_m}{t}.$$

Now, using (3.17) we obtain

$$\frac{-1/p_l + \beta_l(r_l - \delta)}{t} \leq \frac{\sigma'(t)}{\sigma(t)} \leq \frac{-1/p_m + \beta_m(r_m + \delta)}{t}.$$

And due to $0 < \beta_l, \beta_m < 1$ and (3.6) we obtain:

$$(3.19) \quad \frac{r/n - 1/p - \delta}{t} \leq \frac{\sigma'(t)}{\sigma(t)} \leq \frac{r/n - 1/p + \delta}{t},$$

which implies (3.13). Besides, for (3.19) and (3.18)

$$\frac{\beta_j r_j - \delta}{t} \leq \frac{\varphi_j'(\delta_j(t))}{\varphi_j(\delta_j(t))} \delta_j'(t) \leq \frac{\beta_j r_j + \delta}{t}.$$

Then δ_j increases and using this last inequality and (3.17)

$$(3.20) \quad \frac{\beta_j r_j - \delta}{(r_j + \delta)t} \leq \frac{\delta_j'(t)}{\delta_j(t)} \leq \frac{\beta_j r_j + \delta}{(r_j - \delta)t}.$$

From here and (3.8)

$$\frac{\beta_j}{3t} \leq \frac{\delta_j'(t)}{\delta_j(t)} \leq \frac{3\beta_j}{t};$$

and we have proved (3.14). The statement (iv) is the immediate consequence of applying the left inequality of (3.20) and the change of variable $u = \delta_j(t)$.

Finally we multiply (3.12) elevated to $1/r_j$ and use (3.11)

$$\sigma(t)^{n/r} = t^{-n/(rp)} \prod_{j=1}^n \varphi_j(\delta_j(t))^{1/r_j} = t^{-n/(rp)+1} \prod_{j=1}^n \left[\frac{\varphi_j(\delta_j(t))}{\delta_j(t)^{r_j}} \right]^{1/r_j}.$$

So

$$\begin{aligned} \left(\int_0^\infty t^{\theta(1/p-r/n)-1} \sigma(t)^\theta dt \right)^{1/\theta} &= \left(\int_0^\infty \prod_{j=1}^n \left[\frac{\varphi_j(\delta_j(t))}{\delta_j(t)^{r_j}} \right]^{\frac{\theta r}{nr_j}} \frac{dt}{t} \right)^{1/\theta} \leq \\ &\leq \prod_{j=1}^n \left(\int_0^\infty \left[\frac{\varphi_j(\delta_j(t))}{\delta_j(t)^{r_j}} \right]^{\theta_j} \frac{dt}{t} \right)^{\frac{1}{\theta_j} \frac{r}{nr_j}} \end{aligned}$$

and applying (3.15) we obtain (3.10). The lemma is proved. \square

In the following lemma we use the notations (3.5) and (3.6) too.

Lemma 4. *Let $n \in \mathbb{N}$, $0 < r_j < \infty$, $1 \leq p_j < \infty$ and $1 \leq \theta_j \leq \infty$ ($j = 1, \dots, n$) such that $\beta_j > 0$ for all j . Then for every function $f \in S_0(\mathbb{R}^n) \cap L_{loc}(\mathbb{R}^n)$ such that $f \in b_{p_1, \dots, p_n; \theta_1, \dots, \theta_n}^{r_1, \dots, r_n}(\mathbb{R}^n)$*

$$(3.21) \quad f^*(t) \leq (2^{k'} - 1)f^*(\xi t) + c(\xi)\sigma(t), \quad \text{for all } \xi > 1$$

and

$$(3.22) \quad \left(\int_0^\infty t^{\theta(1/p-r/n)-1} \sigma(t)^\theta dt \right)^{1/\theta} \leq \prod_{j=1}^n \|h^{-r_j} w_j^{k_j}(f; h)_{p_j}\|_{\mathcal{L}^{\theta_j}}^{\frac{r}{nr_j}},$$

where $r_j < k_j \in \mathbb{N}$, $k' = \max k_j$.

Proof. First we apply Lemma 1 to f getting its estimation (3.2). Now we define $0 < \delta = \frac{1}{2} \min_j \{\beta_j r_j\}$ and apply Lemma 2 to the modulus of continuity getting

$$w_j^{k_j}(f; u)_{p_j} \leq \varphi_j(u), \quad \varphi_j(u)u^{-r_j+\delta} \uparrow, \quad \varphi_j(u)u^{-r_j-\delta} \downarrow$$

and

$$(3.23) \quad \|\varphi_j(u)u^{-r_j}\|_{\mathcal{L}^{\theta_j}} \leq c \|h^{-r_j} w_j^{k_j}(f; h)_{p_j}\|_{\mathcal{L}^{\theta_j}}.$$

Then we obtain (3.21) with $\sigma(t)$ of (3.9) kind. Only rests to apply Lemma 3 and use (3.23). \square

4. RESULTS

Theorem 1 (embedding with no limit exponent). *Assume that a function f satisfies the conditions of Lemma 4 and $f \in L^1(\mathbb{R}^n) + L^{p_0}(\mathbb{R}^n)$. For some $p_0 > 0$ such that*

$$\frac{1}{p_0} > \frac{1}{p} - \frac{r}{n}.$$

Let $\max(1, p_0) < q < \infty$ and

$$(4.1) \quad \frac{1}{q} > \frac{1}{p} - \frac{r}{n}.$$

Then for any $s > 0$ $f \in L^{q,s}(\mathbb{R}^n)$ and

$$(4.2) \quad \|f\|_{q,s} \leq c \left[\|f\|_{L^1+L^{p_0}} + \prod_{j=1}^n \|f\|_{b_{p_j, \theta_j; j}^{r_j}} \right].$$

Proof. The proof follows the scheme of the one of [8, Corollary 2.5]. We include it here for the reader's convenience. We can assume that $s < \min(1, p_0, \theta)$. Let $f = g + h$, with $g \in L^1(\mathbb{R}^n)$ and $h \in L^{p_0}(\mathbb{R}^n)$. Applying the Hölder inequality, we obtain

$$J_1 \equiv \int_1^\infty [t^{1/q} f^*(t)]^s \frac{dt}{t} \leq c \left[\left(\int_0^\infty g^*(t) dt \right)^s + \left(\int_0^\infty h^*(t)^{p_0} dt \right)^{s/p_0} \right].$$

It follows that

$$(4.3) \quad J_1 \leq c' \|f\|_{L^1+L^{p_0}}^s.$$

Let $0 < \delta < 1$. Using (3.21) with $\xi = (2^{1/s} K)^q$, we get by Hölder inequality and (4.1):

$$\begin{aligned} J_\delta \equiv \int_\delta^\infty [t^{1/q} f^*(t)]^s \frac{dt}{t} &\leq J_1 + K^s \int_\delta^1 [t^{1/q} f^*(\xi t)]^s \frac{dt}{t} + \\ &+ c \int_0^1 t^{s/q-1} \sigma(t)^s dt \leq J_1 + \frac{1}{2} J_\delta + \\ &+ c' \left(\int_0^1 t^{\theta(1/p-r/n)-1} \sigma(t)^\theta dt \right)^{s/\theta}. \end{aligned}$$

The inequality (4.2) follows now from (3.22) and (4.3). \square

Theorem 2 (embedding with limit exponent). *Let $n \in \mathbb{N}$, $0 < r_j < \infty$, $1 \leq p_j < \infty$, $1 \leq \theta_j \leq \infty$. Define r , p and θ as in (3.5) and*

suppose that (see (3.6)) $\beta_j > 0$ ($1 \leq j \leq n$) and $p < \frac{n}{r}$. We define $q^* = np/(n - rp)$. Then, for every function $f \in L_{loc}(\mathbb{R}^n)$ there holds

$$\|f\|_{q^*, \theta} \leq c \prod_{j=1}^n \left[\|f\|_{b_{p_j, \theta_j; j}^{r_j}} \right]^{\frac{r}{nr_j}}$$

This statement follows immediately from Lemma 4. Let's note that the condition of $f \in L_{loc}(\mathbb{R}^n)$ and being of compact support can be substituted by $f \in L^{p_0}(\mathbb{R}^n)$ for some $0 < p_0 < q^*$. In this case we have the embedding

$$L^{p_0} \cap b_{p_1, \dots, p_n; \theta_1, \dots, \theta_n}^{r_1, \dots, r_n}(\mathbb{R}^n) \hookrightarrow L^{q^*, \theta}(\mathbb{R}^n).$$

The following theorems is our main result. It expresses an embedding of different metrics.

Theorem 3. Let $n \in \mathbb{N}$, $0 < r_j < \infty$, $1 \leq p_j < \infty$, $1 \leq \theta_j \leq \infty$ $j \in \{1, \dots, n\}$. Let r , p and θ be the numbers defined in (3.5) and suppose that for all $1 \leq j \leq n$

$$\beta_j = \frac{1}{r_j} \left(\frac{r}{n} + \frac{1}{p_j} - \frac{1}{p} \right) > 0.$$

We choose arbitrary $p_j < q_j < \infty$ such that

$$\frac{1}{q_j} > \frac{1}{p} - \frac{r}{n}$$

and denote

$$\begin{aligned} \varkappa_j &= 1 - \frac{1}{\beta_j r_j} \left(\frac{1}{p_j} - \frac{1}{q_j} \right), \\ \alpha_j &= \varkappa_j r_j \quad \text{and} \quad \frac{1}{\theta'_j} = \frac{1 - \varkappa_j}{\theta} + \frac{\varkappa_j}{\theta_j}. \end{aligned}$$

Then for any function $f \in S_0(\mathbb{R}^n)$ such that $f \in b_{p_1, \dots, p_n; \theta_1, \dots, \theta_n}^{r_1, \dots, r_n}(\mathbb{R}^n)$ there holds the inequality

$$(4.4) \quad \left(\int_0^\infty [h^{-\alpha_j} \|\Delta_j^{r_j}(h)f\|_{q_j, 1}]^{\theta'_j} \frac{dh}{h} \right)^{1/\theta'_j} \leq c \sum_{i=1}^n \|f\|_{b_{p_i, \theta_i; i}^{r_i}}$$

(where $r_j < k_j \in \mathbb{N}$); which implies the embedding

$$(4.5) \quad b_{p_1, \dots, p_n; \theta_1, \dots, \theta_n}^{r_1, \dots, r_n}(\mathbb{R}^n) \hookrightarrow b_{q_1, \dots, q_n; \theta'_1, \dots, \theta'_n}^{\alpha_1, \dots, \alpha_n}(\mathbb{R}^n).$$

Proof. Suppose that $j = 1$. Note that $0 < \varkappa_1 < 1$. Set now $r_i < k_i \in \mathbb{N}$ for every $i = 1, \dots, n$. Denote for $h > 0$ and $x \in \mathbb{R}^n$

$$f_h(x) = |\Delta_1^{k_1}(h)f(x)|.$$

As $\|f_h\|_{p_1} \leq w_1^{k_1}(f; h)_{p_1} < +\infty$, $f_h \in L^{p_1}(\mathbb{R}^n)$ and applying Theorem 1 we have that $f_h \in L^{q_1, 1}(\mathbb{R}^n)$. Denote for $h > 0$

$$J(h) \equiv \|f_h\|_{q_1, 1} = \int_0^\infty t^{1/q_1-1} f_h^*(t) dt < \infty.$$

Set $\xi_0 = (2^{k+2})^{q_1}$ and

$$(4.6) \quad Q(h) = \{t > 0 : f_h^*(t) \geq 2^{k+1} f_h^*(\xi_0 t)\},$$

where $k = \max k_i$ as in Lemma 1. Then

$$\begin{aligned} \int_{\mathbb{R}_+ - Q(h)} t^{1/q_1-1} f_h^*(t) dt &\leq 2^{k+1} \int_0^\infty t^{1/q_1-1} f_h^*(\xi_0 t) dt = \\ &= 2^{k+1} \xi_0^{-1/q_1} \int_0^\infty t^{1/q_1-1} f_h^*(t) dt = \frac{1}{2} J(h). \end{aligned}$$

Therefore

$$J(h) \leq 2 \int_{Q(h)} t^{1/q_1-1} f_h^*(t) dt \equiv 2J'(h).$$

So, it is enough to estimate f_h in $Q(h)$. Now we choose

$$(4.7) \quad 0 < \delta = \frac{1}{2} \min\{\min_i \{\beta_i r_i\}, \frac{1}{q_1} - \frac{1}{p} + \frac{r}{n}\}$$

By virtue of Lemma 2 there exists functions $\varphi_i(t)$ on \mathbb{R}_+ ($i = 1, \dots, n$) continuously differentiable such that

$$(4.8) \quad \varphi_i(t) t^{-r_i-\delta} \downarrow \text{ and } \varphi_i(t) t^{-r_i+\delta} \uparrow;$$

$$(4.9) \quad w_i^{k_i}(f; t)_{p_i} \leq \varphi_i(t);$$

$$(4.10) \quad \|t^{-r_i} \varphi_i(t)\|_{\mathcal{L}^{\theta_i}} \leq c \|t^{-r_i} w_i^{k_i}(f; t)_{p_i}\|_{\mathcal{L}^{\theta_i}} = c \|f\|_{b_{p_i, \theta_i; i}^{r_i}}.$$

Now, applying Lemma 1, (4.9) and (4.6) we have for all $t \in Q(h)$

$$f_h^*(t) \leq c \max_{1 \leq i \leq n} t^{-1/p_i} \varphi_i(\delta_i(t)),$$

with δ_i any functions on \mathbb{R}_+ such that $\prod_{i=1}^n \delta_i(t) = t$.

Due to (4.7) and (4.8) we apply Lemma 3 and we have that there exists a non-negative function $\sigma(t)$ such that

$$(4.11) \quad f_h^*(t) \leq c \sigma(t)$$

$$(4.12) \quad \left(\int_0^\infty t^{\theta(1/p-r/n)-1} \sigma(t)^\theta dt \right)^{1/\theta} \leq \prod_{i=1}^n [\|\varphi_i(t) t^{-r_i}\|_{\mathcal{L}^{\theta_i}}]^{\frac{r}{nr_i}}.$$

There exist positive, continuously differentiable functions $u_i(t)$ on \mathbb{R}_+ such that $\prod_{i=1}^n u_i(t) = t$ and

$$(4.13) \quad \sigma(t) = t^{-1/p_i} \varphi_i(u_i(t)) \quad \forall i = 1, \dots, n.$$

$$(4.14) \quad \sigma(t) t^{1/p-r/n+\delta} \uparrow,$$

$$(4.15) \quad u_1(t) t^{-\frac{\beta_1}{3}} \uparrow \quad \text{and} \quad u_1(t) t^{-3\beta_1} \downarrow,$$

$$(4.16) \quad \left(\int_0^\infty [u_1(t)^{-r_1} \varphi_1(u_1(t))]^{\theta_1} \frac{dt}{t} \right)^{1/\theta_1} \leq c \|\varphi_1(t) t^{-r_1}\|_{\mathcal{L}^{\theta_1}}.$$

The estimation (4.11) can be used for “little” t . For “big” t we will use the following estimate, which is consequence of a weak type inequality and (4.9).

$$(4.17) \quad f_h^*(t) \leq t^{-1/p_1} \|f_h\|_{p_1} \leq t^{-1/p_1} w_1^{k_1}(f; h)_{p_1} \leq t^{-1/p_1} \varphi_1(h).$$

Then

$$(4.18) \quad J'(h) \leq \int_0^\infty t^{1/q_1-1} \Phi(t, h) dt$$

where

$$(4.19) \quad \Phi(t, h) = \min\{\sigma(t), t^{-1/p_1} \varphi_1(h)\}.$$

Then

$$\begin{aligned} J &\equiv \|h^{-\alpha_1} J(h)\|_{\mathcal{L}^{\theta'_1}} \leq \\ &c \left(\|h^{-\alpha_1} \int_{\{h \leq u_1(t)\}} t^{1/q_1-1/p_1-1} \varphi_1(h) dt \|_{\mathcal{L}^{\theta'_1}} + \right. \\ &\quad \left. + \|h^{-\alpha_1} \int_{\{h \geq u_1(t)\}} t^{1/q_1-1} \sigma(t) dt \|_{\mathcal{L}^{\theta'_1}} \right) \equiv c[J_1 + J_2]. \end{aligned}$$

Due to (4.15) u_1 possess positive inverse $u_1^{-1} \equiv \beta$ on \mathbb{R}_+ and

$$(4.20) \quad \frac{u_1'(t)}{u_1(t)} \leq \frac{c}{t}.$$

Now

$$(4.21) \quad J_1 = \|h^{-\alpha_1} \varphi_1(h) \int_{\beta(h)}^\infty t^{1/q_1-1/p_1-1} dt \|_{\mathcal{L}^{\theta'_1}} = c \|h^{-\alpha_1} \varphi_1(h) \beta(h)^{(1/q_1-1/p_1)} \|_{\mathcal{L}^{\theta'_1}}.$$

Now we proceed with J_2 . Due to (4.14) and (4.7)

$$\begin{aligned} J_2 &= \|h^{-\alpha_1} \int_0^{\beta(h)} t^{1/q_1-1} \sigma(t) dt\|_{\mathcal{L}^{\theta'_1}} \leq \\ &\leq \|h^{-\alpha_1} \beta(h)^{1/p-r/n+\delta} \sigma(\beta(h)) \int_0^{\beta(h)} t^{1/q_1-1/p+r/n-\delta-1} dt\|_{\mathcal{L}^{\theta'_1}} = \\ &= c \|h^{-\alpha_1} \beta(h)^{1/q_1} \sigma(\beta(h))\|_{\mathcal{L}^{\theta'_1}} \end{aligned}$$

By (4.13) the last integral is the same as one in the right hand of (4.21). Therefore we have that

$$J \leq c \|h^{-\alpha_1} \beta(h)^{\frac{1}{q_1}-\frac{\varkappa_1}{p_1}} \varphi_1(h)^{\varkappa_1} \sigma(\beta(h))^{(1-\varkappa_1)}\|_{\mathcal{L}^{\theta'_1}}.$$

The change of variable $u_1(z) = h$ and (4.20) arrive us to

$$J \leq c \|u_1(z)^{-\alpha_1} \varphi_1(u_1(z))^{\varkappa_1} z^{\frac{1}{q_1}-\frac{\varkappa_1}{p_1}} \sigma(z)^{(1-\varkappa_1)}\|_{\mathcal{L}^{\theta'_1}}.$$

And using Hölder's inequality with exponents $u = \theta_1/(\varkappa_1\theta'_1)$ and $u' = \theta_1/(\theta_1 - \varkappa_1\theta'_1)$ (observe that $(1 - \varkappa_1)\theta'_1 u' = \theta$, $(\frac{\theta'_1}{q_1} - \frac{\varkappa_1\theta'_1}{p_1})u' = \theta(\frac{1}{p} - \frac{r}{n})$).

$$J \leq \left(\int_0^\infty \left[\frac{\varphi_1(u_1(z))}{u_1(z)^{r_1}} \right]^{\theta_1} \frac{dz}{z} \right)^{\varkappa_1/\theta_1} \left(\int_0^\infty z^{\theta(1/p-r/n)} \sigma(z)^\theta \frac{dz}{z} \right)^{(1-\varkappa_1)/\theta}.$$

From here and (4.16), (4.12) and (4.10) follows

$$J \leq (\|t^{-r_1} w_1^{k_1}(f; t)_{p_1}\|_{\mathcal{L}^{\theta_1}})^{\varkappa_1} \prod_{i=1}^n (\|t^{-r_i} w_i^{k_i}(f; t)_{p_i}\|_{\mathcal{L}^{\theta_i}})^{\frac{r(1-\varkappa_1)}{nr_i}}.$$

which, using the inequality between arithmetic and geometric means, implies (4.4). The theorem is proved. \square

Remark 1. Let us note that in (4.4) it appears the stronger Lorentz norm $L^{q_j,1}$ instead of the norm L^{q_j} . Then (4.4) is stronger than (4.5). A detailed reading of the proof shows that in fact, in the right part of (4.4) can appear the $L^{q_j,\xi}$ norm, for every $\xi > 0$. Note that in this case “ c ” in the right part of (4.4) explodes when ξ goes to 0.

Remark 2. The values of the parameters θ'_j found in Theorem 3 are sharp. In order to see that the values of θ'_j can not be smaller one can consider the close embedding relation between Sobolev and Besov spaces and the fact that the parameters found in [8] are sharp ([8, Remark 3.3]).

REFERENCES

- [1] BENNETT, C., and SHARPLEY, R.: Interpolation of Operators, Academic Press, 1988.
- [2] BESOV, O.V., IL'IN, V.P., and NIKOL'SKIĬ, S.M.: Integral Representation of Functions and Imbedding Theorems, vol. 1 – 2, Winston, Washington D.C., Halsted, New York–Toronto–London, 1978
- [3] GOLOVKIN, K.K.: A generalization of Marcinkiewicz's interpolation theorem, Trudy Mat. Inst. Steklov **102** (1967), 5 – 28.
- [4] HERZ, C.: Lipschitz spaces and Bernstein's theorem of absolutely convergent Fourier transform, J. Math. Mech. **18** No. 18 (1968), 283 – 323.
- [5] KOLYADA, V.I.: On embedding of Sobolev spaces, Mat. Zametki **54** No. 3 (1993), 48 – 71; English transl. in Math. Notes **54** No. 3 (1993), 908 – 922
- [6] KOLYADA, V.I.: Rearrangements of functions and embedding of anisotropic spaces of the Sobolev type, East J. on Approximations **4** No. 2 (1998), 111 – 199.
- [7] KOLYADA, V.I.: Embeddings of fractional Sobolev spaces and estimates of Fourier transforms, Mat. Sb. **192** No. 7 (2001), 51 – 72; English transl. in Sbornik: Mathematics **192** No. 7 (2001), 979 – 1000
- [8] KOLYADA, V.I., AND PÉREZ, F.J.: Estimates of difference norms for functions in anisotropic Sobolev spaces, Mat. Nachr. **267**, 46–64 (2004).
- [9] NIKOL'SKIĬ, S.M.: Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of several variables, Trudy Mat. Inst. Steklov **38** (1951), 244 – 278 (in Russian).
- [10] NIKOL'SKIĬ, S.M.: Approximation of Functions of Several Variables and Imbedding Theorems, Springer – Verlag, Berlin – Heidelberg – New York, 1975
- [11] PEETRE, J.: Espaces d'interpolation et espaces de Soboleff, Ann. Inst. Fourier (Grenoble) **16** (1966), 279 – 317
- [12] PÉREZ LÁZARO, F.J.: A note on extreme cases of Sobolev embeddings, J. Math. Anal. Appl. **320** (2006) 973 – 982.
- [13] UL'YANOV, P.L.: On the embedding of certain classes of functions, Mat. Zametki **1** 4(1967), 405–414.

DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, EDIFICIO J. L. VIVES, CALLE LUIS DE ULLOA S/N, 26004 LOGROÑO, SPAIN

E-mail address: francisco-javier.per@dmc.unirioja.es